Chapter 5

Univariate Time Series: Volatility Models

5.1   Introduction

In Chapter 3 we have considered approaches to modelling conditional mean of a univariate time series. However, many areas of financial theory are concerned with the second moment of time series – conditional volatility as a proxy for risk.

In this chapter we introduce time series models that represent the dynamics of conditional variances. In particular we consider ARCH, GARCH model as well as their extensions.

The reader is also referred to Engle (1982), Bollerslev (1986), Nelson (1991), Hamilton (1994), Enders (2004), Zivot and Wang (2006).

5.2   The ARCH Model

Besides a time varying conditional mean of financial time series, most of them also exhibit changes in volatility regimes. This is especially applicable to many high frequency macroeconomic and financial time series.

While modelling such time series, we cannot use homoscedastic models. The simplest way to allow volatility to vary is to model conditional variance using a simple autoregressive (AR) process.

Let Y_t denote a stationary time series, then Y_t can be expressed as its mean plus a white noise:

$$Y_t = c + u_t \tag{5.2.1}$$

where c is the mean of Y_t, and u_t is i.i.d. with mean zero. To allow for conditional

heteroscedasticity, assume that

$$var_{t-1}[u_t] = \sigma_t^2.$$

Here var_{t-1} denotes the variance conditional on information at time $t-1$, and is modelled in the following way:

$$\sigma_t^2 = \alpha_0 + \alpha_1 u_{t-1}^2 + ... + \alpha_p u_{t-p}^2. \qquad (5.2.2)$$

In order to show that this specification is equivalent to AR representation of squared residuals, note that $var_{t-1}[u_t] = E\left[u_{t-1}^2\right] = \sigma_t^2$ since $E[u_t] = 0$. Thus, equation (5.2.2) can be rewritten as:

$$u_t^2 = \alpha_0 + \alpha_1 u_{t-1}^2 + ... + \alpha_p u_{t-p}^2 + \varepsilon_t, \qquad (5.2.3)$$

where $\varepsilon_t = u_t^2 - E_{t-1}\left[u_t^2\right]$ is a zero mean white noise process. The model in (5.2.1) and (5.2.3) is known as the autoregressive conditional heteroscedasticity (ARCH) model of Engle (1982), which is usually referred to as the $ARCH(p)$ model. More generally, ARCH model can be rewritten as

$$Y_t = c + u_t$$
$$u_t = \sigma_t \eta_t$$
$$\sigma_t^2 = \alpha_0 + \alpha_1 u_{t-1}^2 + ... + \alpha_p u_{t-p}^2,$$

where η_t is an iid normal random variable.

5.2.1   Example: Simulating an $ARCH(p)$ model in EViews

It is relatively easy to simulate ARCH process in EViews. Let us consider as example the following $ARCH(2)$ model

$$Y_t = \sigma_t \eta_t$$
$$\sigma_t^2 = 3.5 + 0.5 Y_{t-1}^2 + 0.48 Y_{t-2}^2 \qquad (5.2.4)$$

with η_t being independent random variables following $N(0,1)$ distribution. Similarly to ARMA process we need to generate error term process η_t and first two initial values of Y_t after which the whole process can be simulated. Create a new workfile and in the command line enter
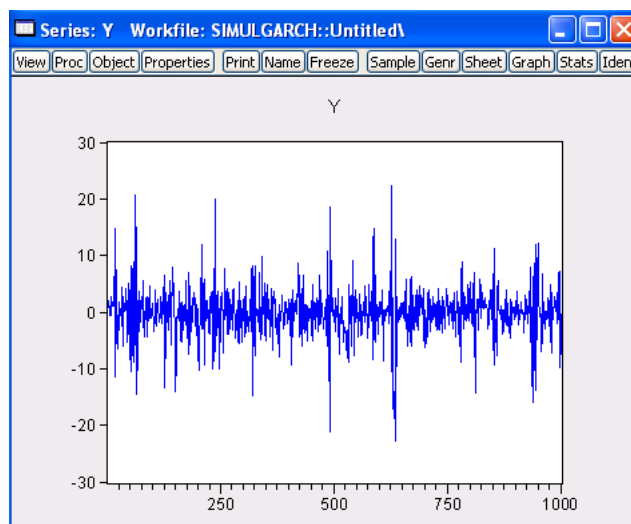
    smpl @all

    series eta=nrnd

    smpl @first @first+1

series y=@sqrt(3.5/(0.5+0.48))*eta

smpl @first+2 @last

y=@sqrt(3.5+0.5*y(-1)^2+0.48*y(-2)^2)*eta

smpl @all

The last statement is included to ensure that we come back the whole data range. The plot of the simulated series is given in the following figure.

Figure 5.1: Plot of simulated ARCH process

Visually, the process looks stationary, mean reverting and with zero mean as expected from the equation (5.2.4).

Testing for ARCH Effects In order to test for the presence of ARCH effects in the residuals, we can use AR representation of squared residuals in the following way. Based on equation (5.2.2), construct an auxiliary regression

$$\hat{u}_t^2 = \alpha_0 + \alpha_1 \hat{u}_{t-1}^2 + ... + \alpha_p \hat{u}_{t-p}^2 + \varepsilon_t, \qquad (5.2.5)$$

. The significance of parameters α_i would indicate the presence of conditional volatility. Under the null hypothesis that there are no ARCH effects:

$$\alpha_1 = \alpha_2 = ... = \alpha_p = 0,$$

the test statistic $LM = TR^2 \overset{a}{\sim} \chi_p^2$ where T is the sample size and R^2 is computed from the regression (5.2.5).

5.3   The GARCH Model

More general form of conditional volatility is based on ARMA specification as an extension of AR process of squared residuals. Bollerslev (1986) introduces GARCH model (which stands for generalized ARCH) where he replaces the AR model in (5.2.2) by:

$$\sigma_t^2 = \alpha_0 + \sum_{i=1}^{p} \alpha_i u_{t-i}^2 + \sum_{j=1}^{q} \beta_i \sigma_{t-j}^2, \qquad (5.3.1)$$

where the coefficients α_i and β_j are positive to ensure that the conditional variance σ_t^2 is always positive. In order to emphasize the number of lags used in (5.3.1) we denote the model by $GARCH(p,q)$.

When $q = 0$, the GARCH model reduces to the ARCH model. Under the $GARCH(p,q)$ model, the conditional variance of u_t, σ_t^2, depends on the squared residuals in the previous p periods, and the conditional variance in the previous q periods. The most commonly used model is a $GARCH(1,1)$ model with only three parameters in the conditional variance equation.

A GARCH model can be expressed as an ARMA model of squared residuals. For example, for a $GARCH(1,1)$ model:

$$\sigma_t^2 = \alpha_0 + \alpha_1 u_{t-1}^2 + \beta_1 \sigma_{t-1}^2.$$

Since $E_{t-1}\left[u_t^2\right] = \sigma_t^2$, the above equation can be rewritten as:

$$u_t^2 = \alpha_0 + (\alpha_1 + \beta_1)u_{t-1}^2 + \varepsilon_t - \beta_1 \varepsilon_{t-1}, \qquad (5.3.2)$$

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which is an $ARMA(1,1)$ model. Here $\varepsilon_t = u_t^2 - E_{t-1}\left[u_t^2\right]$ is the white noise error term.

Given the ARMA representation of the GARCH model, we conclude that stationarity of the $GARCH(1,1)$ model requires $\alpha_1 + \beta_1 < 1$. The unconditional variance of u_t is given by

$$var[u_t] = E\left[u_t^2\right] = \alpha_0/(1 - \alpha_1 - \beta_1),$$

Indeed, from (5.3.2)

$$E\left[u_t^2\right] = \alpha_0 + (\alpha_1 + \beta_1)E\left[u_{t-1}^2\right]$$

and thus $E\left[u_t^2\right] = \alpha_0 + (\alpha_1 + \beta_1)E[u_t^2]$ since u_t^2 is stationary. For the general $GARCH(p,q)$ model (5.3.2), the squared residuals u_t^2 behave like an $ARMA(\max(p,q),q)$ process.

One can identify the orders of the GARCH model using the correlogram of the squared residuals. They will coincide with ARMA orders of the squared residuals of the time series.

GARCH Model and Stylized Facts In practice, researchers have uncovered many so-called stylized facts about the volatility of financial time series; Bollerslev, Engle and Nelson (1994) give a complete account of these facts. Using the ARMA representation of GARCH models shows that the GARCH model is capable of explaining many of those stylized facts. This section will focus on three important ones: volatility clustering, fat tails, and volatility mean reversion. Other stylized facts are illustrated and explained in later sections.

Volatility Clustering Usually the GARCH coefficient β_1 is found to be around 0.9 for many weekly or daily financial time series. Given this value of β_1, it is obvious that large values of σ_{t-1}^2 will be followed by large values of σ_t^2, and small values of σ_{t-1}^2 will be followed by small values of σ_t^2. The same reasoning can be obtained from the ARMA representation in (5.3.2), where large/small changes in u_{t-1}^2 will be followed by large/small changes in σ_t^2.

Fat Tails It is well known that the distribution of many high frequency financial time series usually have fatter tails than a normal distribution. This means that large changes are more often to occur than under a normal distribution. Thus a GARCH model can replicate the fat tails usually observed in financial time series.

Volatility Mean Reversion Although financial markets may experience excessive volatility from time to time, it appears that volatility will eventually settle down to a long run level. The previous subsection showed that the long run variance of u_t for the stationary $GARCH(1,1)$ model is $\alpha_0/(1 - \alpha_1 - \beta_1)$. In this case, the volatility is always pulled toward this long run level.

5.3.1   Example: Simulating an $GARCH(p,q)$ model in EViews

It is slightly trickier to simulate GARCH process than the ARCH one in EViews. Since it is necessary simultaneously to generate Y_t and σ_t processes, we will need to use loop to accomplish it. Therefore, it is more convenient to use program object rather than doing it in the command line. Consider as an example $GARCH(2,1)$ series

$$Y_t = \sigma_t \eta_t$$
$$\sigma_t^2 = 3.5 + 0.5Y_{t-1}^2 + 0.28Y_{t-2}^2 + 0.2\sigma_{t-1}^2 \tag{5.3.3}$$

We start the program with the same commands as in the ARCH case; the only difference is that we generate a conditional variance process s.

    smpl @all

    series eta=nrnd

    scalar n=@obs(eta)

    smpl @first @first+1

    series s=3.5/(0.5+0.28+0.2)
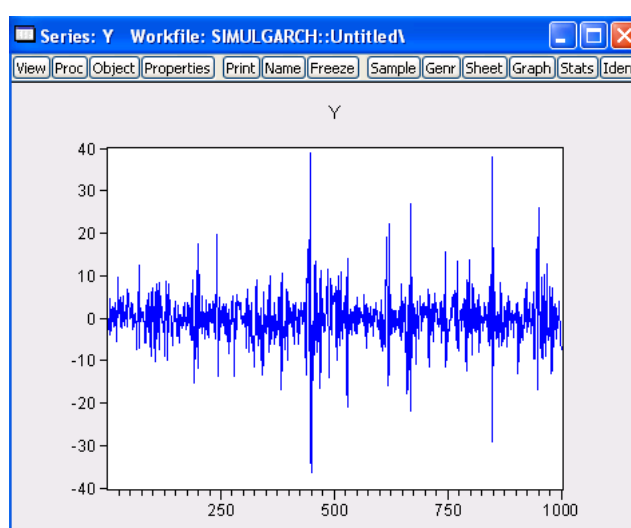
    series y=@sqrt(s)*eta

The next part of the program creates the loop where both series Y_t and σ_t^2 are generated observation after observation.

```
for !i=2 to n-2

smpl @first+!i @first+!i

s=3.5+0.5*y(-1)^2+0.28*y(-2)^2+0.2*s(-1)

y=@sqrt(s)*eta

next

smpl @all
```

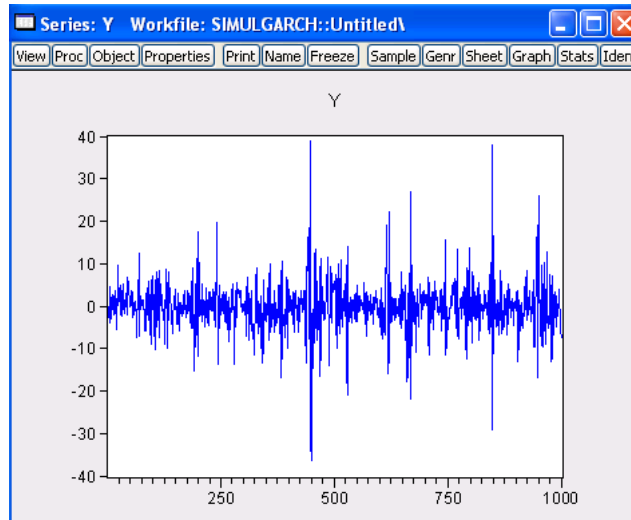The graph of the simulated GARCH process is given on Figure **??**.

Figure 5.2: Plot of the simulated GARCH process

We can see on the graph a clear effect of volatility clustering. In most cases volatility stays low but there are several spikes with high volatility which persist for a number of periods. Another stylized fact can be seen from the histogram of the simulated observations (click on **View/Descriptive Statistic and Tests/Histogram and Stats**). Jarque-Bera test strongly rejects the null hypothesis of normality and the kurtosis is extremely high indicating fat tails of the generated distribution.

5.4   GARCH model estimation

This section illustrates how to estimate a GARCH model. Assuming that u_t follows normal or Gaussian distribution conditional on past history, the prediction error

Figure 5.3: Histogram of the simulated GARCH process

decomposition of the log-likelihood function of the GARCH model conditional on initial values is:

$$\log L = -\frac{T}{2}\log(2\pi) - \frac{1}{2}\sum_{i=1}^{T}\log\left(\sigma_t^2\right) - \frac{1}{2}\sum_{i=1}^{T}\frac{u_t^2}{\sigma_t^2}.$$

The unknown model parameters c, α_i $(i = 0, ..., p)$ and β_j, $(j = 1, ..., q)$ can be estimated using conditional maximum likelihood estimation (MLE). Details of the maximization are given in Hamilton (1994). Once the MLE estimates of the parameters are found, estimates of the time varying volatility σ_2 $(t = 1, ..., T)$ are also obtained as a side product.

5.5   GARCH Model Extensions

In many cases, the basic GARCH model (5.3.2) provides a reasonably good model for analyzing financial time series and estimating conditional volatility. However, there are some aspects of the model which can be improved so that it can better capture the characteristics and dynamics of a particular time series.

In the basic GARCH model, since only squared residuals u_{t-i}^2 enter the equation, the signs of the residuals or shocks have no effects on conditional volatility. However, a stylized fact of financial volatility is that bad news (negative shocks) tends to have a larger impact on volatility than good news (positive shocks).

5.5.1   EGARCH Model

Nelson (1991) proposed the following exponential GARCH (EGARCH) model to allow for leverage effects:

$$h_t = \alpha_0 + \sum_{i=1}^{p} \alpha_i \frac{|u_{t-i}| + \gamma_i u_{t-i}}{\sigma_{t-i}} + \sum_{i=1}^{q} h_{t-j},$$

where $h_t = \log \sigma_t^2$. Note that when u_{t-i} is positive, the total effect of u_{t-i} is $(1 + \gamma_i)|u_{t-i}|$; in contrast, when u_{t-i} is negative, the total effect of u_{t-i} is $(1 - \gamma_i)|u_{t-i}|$. Bad news can have a larger impact on volatility, and the value of γ_i would be expected to be negative.

5.5.2   TGARCH Model

Another GARCH variant that is capable of modeling leverage effects is the threshold GARCH (TGARCH) model, which has the following form:

$$\sigma_t^2 = \alpha_0 + \sum_{i=1}^{p} \alpha_i u_{t-i}^2 + \sum_{i=1}^{p} \alpha_i S_{t-i} u_{t-i}^2 + \sum_{j=1}^{q} \beta_j \sigma_{t-j}^2,$$

where

$$S_{t-i} = \begin{cases} 1 & u_{t-i} < 0 \\ 0 & u_{t-i} \geq 0 \end{cases}.$$

That is, depending on whether u_{t-i} is above or below the threshold value of zero, u_{t-i}^2 has different effects on the conditional variance σ_t^2: when u_{t-i} is positive, the total effects are given by $\alpha_i u_{t-i}^2$; when u_{t-i} is negative, the total effects are given by $(\alpha_i + \gamma_i)u_{t-i}^2$. So one would expect γ_i to be positive for bad news to have larger impacts.

5.5.3   PGARCH Model

The basic GARCH model can be also extended to allow for leverage effects. This is made possible by treating the basic GARCH model as a special case of the power GARCH (PGARCH) model proposed by Ding and (1993) (1993):

$$\sigma_t^d = \alpha_0 + \sum_{i=1}^{p} \alpha_i \left(|u_{t-i}| + \gamma_i u_{t-i}\right)^d + \sum_{j=1}^{q} \beta_j \sigma_{t-j}^d, \qquad (5.5.1)$$

where d is a positive exponent, and γ_i denotes the coefficient of leverage effects. Note that when $d = 2$, (5.5.1) reduces to the basic GARCH model with leverage effects.

Two Components Model The GARCH model can be used to model mean reversion in conditional volatility. Recall the mean reverting form of the basic $GARCH(1,1)$ model:

$$(u_t^2 - \bar{\sigma}^2) = (\alpha_1 + \beta_1)(u_{t-1}^2 - \bar{\sigma}^2) + \varepsilon_t - \beta_1 \varepsilon_{t-1},$$

where $\bar{\sigma}^2 = \alpha_0/(1 - \alpha_1 - \beta_1)$ is the unconditional long run level of volatility which is constant over time. Engle and Lee (1999) propose a model with time varying long run volatility level. The general form of the two components model is:

$$\sigma_t^2 = q_t + s_t \qquad (5.5.2)$$
$$q_t^d = \alpha_1 |u_{t-1}|^d + \beta_1 q_{t-1}^d \qquad (5.5.3)$$
$$s_t^d = \alpha_0 + \alpha_2 |u_{t-1}|^d + \beta_2 s_{t-1}^d. \qquad (5.5.4)$$

The long run component q_t follows a highly persistent $PGARCH(1,1)$ model, and the transitory component s_t follows another $PGARCH(1,1)$ model.

GARCH-in-the-Mean Model In financial investment, high risk is often expected to lead to high returns. Although modern capital asset pricing theory does not imply such a simple relationship, it does suggest there are some interactions between expected returns and risk as measured by volatility. Engle, Lilien and Robins (1987) propose to extend the basic GARCH model so that the conditional volatility can generate a risk premium which is part of the expected returns. This

extended GARCH model is often referred to as GARCH-in-the-mean (GARCH-M) model.

The GARCH-M model extends the conditional mean equation (5.2.1) as follows:

$$Y_t = c + \alpha g(\sigma_t) + u_t,$$

where $g(\cdot)$ can be an arbitrary function of volatility σ_t.

Exogenous Variables in Conditional Mean So far the conditional mean equation has been restricted to a constant when considering GARCH models, except for the GARCH-M model where volatility was allowed to enter the mean equation as an explanatory variable. It is possible to add ARMA terms as well as exogenous explanatory variables in the conditional mean equation. A more general form for the conditional mean equation is

$$Y_t = c + \delta' X_t + u_t$$

where X_t is a $k \times 1$ vector of regressors and δ is a vector of coefficients.

Also, one can add explanatory variables into the conditional variance formula which may have impacts on conditional volatility.

Error Distributions In all the examples illustrated so far, a normal error distribution has been exclusively used. However, given the well known fat tails in financial time series, it may be more desirable to use a distribution which has fatter tails than the normal distribution.

EViews allows two fat-tailed error distributions for fitting GARCH models: the Student t distribution and the generalized error distribution.

5.5.4   Prediction

GARCH models are frequently used to forecast volatility of return. It is straightforward to forecast the conditional variance from an ARCH model. Assuming that the model parameters are known, the one-period ahead forecast is

$$\sigma^2_{t+1|t} = \alpha_0 + \alpha_1 u_t^2 + ... + \alpha_p u_{t-p+1}^2$$

Forecasting the conditional volatility for h periods ahead can be done by a recursion

$$\sigma^2_{t+h|t} = \alpha_0 + \alpha_1 \sigma^2_{t+h-1|t} + ... + \alpha_p \sigma^2_{t+h-p},$$

where $\sigma^2_{t+j} = u^2_{t+j}$ for $j \leq 0$.

The h-period ahead variance forecast for a $GARCH(1,1)$ model is

$$\sigma^2_{t+1|t} = \alpha_0 \left[\sum_{i=0}^{h-1} (\alpha_1 + \beta_1)^i \right] + (\alpha_1 + \beta_1)^h \sigma_t^2.$$

5.5.5 Example: GARCH Estimation

As an example of GARCH model estimation in EViews we consider a series of 2 minutes exchange rates between the Euro and the British Pound for 21 August 2007 between 7:00 and 16:00 GMT. The data is contained in the file EURGBP.wf1. Plot of the EUR/GBP returns is given in Figure **??**.
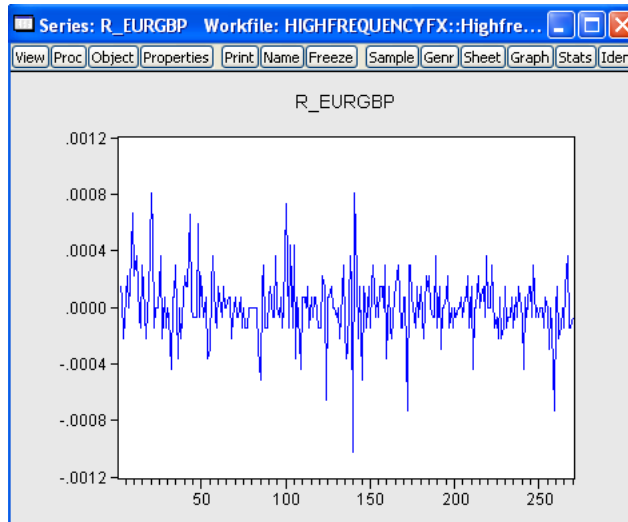

Figure 5.4: 2 minutes EUR/GBP returns on 21 August 2007

We can clearly see periods of high and low volatility of the returns, thus an ARCH type model can be appropriate to model volatility.

Let us first estimate an OLS regression of returns on a constant term. This will give us an opportunity to test for the presence of ARCH effect more formally. Having typed

ls r_eurgbp c

in the command line and pressed **Enter**, go to **View/Residual Tests/ Heteroscedasticity Tests...** in the equation object window and choose **ARCH** option there. The test result is given in Figure **??**

Figure 5.5: ARCH test results

The p-value of the test is very small which rejects the null hypothesis of homoscedasticity of residuals in favor of ARCH alternative. Thus, based on this result we decide to estimate regression with ARCH specification. Go to **Quick/Estimate Equation** option of the main workfile menu and specify the model as you were specifying it for the OLS regression. That is, type r_eurusd c in the **Equation Specification** box. Now, in the **Method** field, choose **ARCH − Autoregressive Conditional Heteroscedasticity**. This will open you more option for ARCH model specification. In the **ARCH-M** we can indicate whether we want to include *ARCH*-in-mean term in the equation and, if yes, whether variance or standard deviation should enter it. In the **Variance and distribution specification** part we can select between simple ARCH/GARCH/TARCH model, EGARCH, PGARCH

or two component GARCH model. We will stay with the simplest first option. In the **Order** field we should write orders of ARCH and GARCH terms in the variance equation. Let us specify GARCH(3,3) model, so enter 3 and 3 respectively in each of the box. If we do not want to include GARCH terms, simply put 0 in front of the **GARCH** field. **Variance regressors** box will be useful if we want to include some exogenous variables in the variance equation. **Errors distribution** box allows us to choose between Gaussian and Student - distributions of the error term. The model estimation output is given in Figure **??**.

Dependent Variable: R_EURGBP
Method: ML - ARCH (Marquardt) - Normal distribution
Date: 01/09/09   Time: 12:51
Sample (adjusted): 2 270
Included observations: 269 after adjustments
Convergence achieved after 198 iterations
Presample variance: backcast (parameter = 0.7)
GARCH = C(2) + C(3)*RESID(-1)^2 + C(4)*RESID(-2)^2 + C(5)*RESID(
    -3)^2 + C(6)*GARCH(-1) + C(7)*GARCH(-2) + C(8)*GARCH(-3)

	Coefficient	Std. Error	z-Statistic	Prob.
C	1.99E-05	1.43E-05	1.390795	0.1643

Variance Equation

	Coefficient	Std. Error	z-Statistic	Prob.
C	2.94E-08	7.73E-09	3.809684	0.0001
RESID(-1)^2	0.049098	0.046037	1.066483	0.2862
RESID(-2)^2	-0.056776	0.051782	-1.096441	0.2729
RESID(-3)^2	0.177889	0.056450	3.151275	0.0016
GARCH(-1)	0.520865	0.274545	1.897193	0.0578
GARCH(-2)	0.113259	0.317913	0.356259	0.7216
GARCH(-3)	-0.349023	0.159471	-2.188629	0.0286

R-squared	-0.001633	Mean dependent var	1.07E-05
Adjusted R-squared	-0.028496	S.D. dependent var	0.000229
S.E. of regression	0.000232	Akaike info criterion	-13.91588
Sum squared resid	1.41E-05	Schwarz criterion	-13.80897
Log likelihood	1879.686	Hannan-Quinn criter.	-13.87295
Durbin-Watson stat	2.149537		

Figure 5.6: GARCH estimation output

The resid terms of the output correspond to α_i coefficients (ARCH terms) and GARCH terms correspond to β_i coefficients in (5.3.2).

We can see that α_3 and β_3 coefficients are statistically significant and α_2 is on a border line of significance.

In **View/Representation** section one can find the variance specification. Also EViews allow to plot both standardized and on-standardized residual plots (in **Actual, Fitted, Residuals**), test for parameter constancy, linear restriction, build correlogram of residuals and squared residuals in the same way as it is done for the OLS regression.

In order to estimate the above model using the command line one should type

equation e.arch(3,3) r_eurgbp c

where the term arch indicates that an ARCH estimation method should be used, order of the ARCH and GARCH terms follow in parentheses. Then you should specify the conditional mean equation as it is done on the least squares model case (the dependent variable should be in the first place).